AFOSR 817

IMM-NYU 290
JUNE 1961

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

Trial and Error Predicates and the Solution to a Problem of Mostowski's

HILARY PUTNAM

PREPARED UNDER
CONTRACT NO. AF49(638)-777
MATHEMATICAL SCIENCES DIRECTORATE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

IMM-NYU 290

New York University
Institute of Mathematical Sciences
Mathematical Sciences Directorate
Air Force Office of Scientific Research
Washington 25, D.C.
AFOSR 817

TRIAL AND ERROR PREDICATES AND THE SOLUTION TO A PROBLEM OF MOSTOWSKI'S

Hilary Putnam

June 8, 1961 Contract No. AF 49(638)-777

ABSTRACT: It is proved that every consistent formula of quantification theory has a model in Mostowski's field of sets.

The research reported in this document has been sponsored by the Mathematical Sciences Directorate, Air Force Office of Scientific Research, under Contract No. AF 49(638)-777.

"TRIAL AND ERROR" PREDICATES AND THE SOLUTION TO A PROBLEM OF MOSTOWSKI'S

By HILARY PUTNAM

1. Introduction. The purpose of this paper is to present two groups of results which have turned out to have a surprisingly close connection. The first two results (Theorems 1 and 2) were inspired by the following question: we know what sets are "decidable" ---namely, the recursive sets (according to Church's Thesis). But what happens if we modify the notion of a decision procedure by (1) allowing the procedure to "change its mind" any finite number of times (in terms of Turing Machines: we visualize the machine as "printing out" a finite sequence of "yesses" and "nos". The last "yes" or "no" is always to be the correct answer); and (2) we give up the requirement that it be possible to tell (effectively) if the computation has terminated? (i.e. if the machine has most recently printed "yes", then we know that the appropriate number must be in the set unless the machine "changes its mind"; but we have no general procedure for telling whether the machine will "change its mind" or not.)

In traditional philosophic parlance, the sets for which there exist a "decision method" in this widened sense are decidable by "empirical" means, or by using "Humean induction"---for, if we always "bet" that the most recently generated answer is correct, we will make a finite number of mistakes, but we will eventually get (and "stick to") the correct answer. Note, however, that even if we have gotten to the correct answer (the

end of the finite sequence) we are never <u>sure</u> that we have the correct answer. The sense in which this is "Humean Induction" may be appreciated by comparing these remarks with the remarks on "Induction" (in the empirical sciences) in the writings of philosophers of science.

Instead of requiring that the sequence of "yesses" and "nos" be finite and non-empty, we may require that it should always be <u>infinite</u>, but that it should consist entirely of "yesses" (or entirely of "nos") from a certain point on: the class of predicates obtained (which we call the class of "trial and error" predicates, for reasons which should be obvious from the foregoing remarks) is easily seen to be unchanged[1]. We thus arrive at the following reformulation of our first question: First define----P is a <u>trial and error predicate</u> if and only if there is a (general recursive) function f such that (for every $x_1, x_2, \ldots, x_n$)

$$P(x_1, \ldots, x_n) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_n, y) = 1$$

$$\overline{P}(x_1, \ldots, x_n) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_n, y) = 0$$

where

$$\lim_{y \to \infty} f(x_1, \ldots, x_n, y) = k =_{df} (Ey)(z)(z \geq y \Rightarrow f(x_1, \ldots, x_n, z) = k)$$

Then we ask

Question 1: What are necessary and sufficient conditions (in terms of the Kleene-Post Hierarchy of arithmetic predicates) that P be a trial and error predicate?

set of all finite sequences) [illegible] never give [illegible] the correct answer. [illegible] "inductive" may be [illegible] by comparing [illegible] the concept of "Induction" in the [illegible] literature in the [illegible] of philosophers of science.

Instead of requiring that the [illegible] "finite" and "must be finite and [illegible]", [illegible] that it should always be infinite, but that it should [illegible] of "guesses" [illegible] [illegible] of "[illegible]" [illegible] [illegible] [illegible] [illegible] [illegible] (which [illegible] of [illegible] and "[illegible]" [illegible]. For reasons which should be [illegible] [illegible] [illegible] is [illegible] to be [illegible][1]. [illegible] the following [illegible] of [illegible] "[illegible]" [illegible] as a limit and [illegible] [illegible] [illegible] [illegible] function f such that (for every $x_1, \dots, x_n$)

$$f(x_1,\dots,x_n) = \lim_{y\to\infty} F(x_1,\dots,x_n,y)$$

$$\bar{f}(x_1,\dots,x_n) = \lim_{y\to\infty} \bar{F}(x_1,\dots,x_n,y)$$

[illegible]

$$\lim_{y\to\infty} F(x_1,\dots,x_n,y) = [illegible]$$

$$F(x_1,\dots,x_n,y) = [illegible]$$

[illegible]

[illegible] "[illegible]" [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

It is obviously better if we know, not just that P is a trial and error predicate, but that (by using an optimal program) we can keep our machine from ever having to change its mind more than k times (for some fixed k, independent of the particular $x_1, \ldots, x_n$ about which we are asking). To make this precise, call a predicate P a <u>k-trial predicate</u> if there is a g.r. function f and a fixed integer k such that (for all $x_1, \ldots, x_n$)

(1) $P(x_1, \ldots, x_n) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_n, y) = 1$

(2) There are <u>at most</u> k integers y such that

$f(x_1, \ldots, x_n, y) \neq f(x_1, \ldots, x_n, y+1)$

[Note that we do not require the function f to be such that

$$\overline{P}(x_1, \ldots, x_n) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_n, y) = 0$$

----however, this condition will always be satisfied as well if we replace the given function f by f^*, where $f^*(x_1, \ldots, x_n, y) = 1$ if $f(x_1, \ldots, x_n, y) = 1$ and $f^*(x_1, \ldots, x_n, y) = 0$ if $f(x_1, \ldots, x_n, y) \neq 1$. For, since there are at most k places (values of y) at which $f(x_1, \ldots, x_n, y)$ changes its value (for fixed $x_1, \ldots, x_n$), there must be a value of y, say M (depending on $x_1, \ldots, x_n$), such that for $y > M$, $f(x_1, \ldots, x_n, y)$ is constant. Then $\lim_{y \to \infty} f(x_1, \ldots, x_n, y) = f(x_1, \ldots, x_n, M+1) \neq 1$ unless $P(x_1, \ldots, x_n)$, so $\overline{P}(x_1, \ldots, x_n) \equiv \lim_{y \to \infty} f^*(x_1, \ldots, x_n, y) = 0$.]

Question 2: What are necessary and sufficient conditions that there exist a k such that P is a k-trial predicate?

The investigations which resolved these questions have led also to other questions. For example we have been able to prove (though not straightforwardly) that (for every k) there is a k+1-trial predicate which enumerates the k-trial predicates (with one less argument place). This theorem is a generalization of Dekker's result that the recursive sets are a recursively enumerable family of r.e. sets[2]; for the recursive sets are the 0-trial sets in our sub-hierarchy, and the r.e. predicates are all 1-trial. Our proof uses Dekker's result, together with the theorem in § 4, that the pairs < A,B > of disjoint r.e. sets are a recursively enumerable family of pairs of r.e. sets. This result is in section § 5 of the present paper, together with results on the modulus of oscillation[3] of trial and error predicates: the most difficult result in § 5 is that the trial and error predicates which possess a recursive modulus of oscillation can be enumerated by a single trial and error predicate. These predicates represent perhaps the largest significant class of $\sum_2 \cap \prod_2$ predicates for which there exists a "normal form"---i.e., a recursively enumerable set of expressions such that (a) every predicate in the class is "designated" by one of the expressions in the set; and (b) given any expression in the set one can effectively write down at least one $\sum_2$ expression[4] and at least one $\prod_2$ expression[4] for the predicate it "designates".

Our second result or group of results is connected with the meta-theory of quantification theory. A number of years ago,

The investigations [illegible] extended [illegible]
also to other equations. [illegible]
(though not [illegible]) [illegible]
[illegible]
[illegible] theorem is a generalization of
[illegible]'s result that [illegible]
able family of [illegible]
[illegible]
[illegible] upon [illegible]'s results [illegible]
that the values [illegible] are [illegible]
enumerable family of [illegible]. This result is [illegible]
tion § 5 of the present [illegible]
of [illegible]
result in § 5 is that [illegible]
a [illegible]
trial and [illegible]
the [illegible]
[illegible]
of [illegible]
[illegible]
the [illegible]
[illegible]
[illegible].

The [illegible]
[illegible]

Mostowski[5] reported on his unsuccessful attempts to find a consistent formula of quantification theory with no model in the "smallest field of sets containing the recursively enumerable sets." Since "set" here means sets of n-tuples, what Mostowski wanted is, in our terminology, a formula with no model in which (1) the universe of discourse is the natural numbers; and (2) the predicate letters are all interpreted as r.e. predicates or truth-functions of r.e. predicates.

The main result of § 3 is : a formula of this kind (the kind wanted by Mostowski) does not exist. Every consistent formula of quantification theory does have a model in $\sum_1^*$.[6] The proof uses Theorems 1 and 2, which were discovered as the answers to Question 1 and 2, and the Hilbert-Bernays-Kleene result that every consistent formula of q.t. (quantification theory) has a model in $\sum_2 \cap \prod_2$. In 1957[8] I gave an example of a consistent formula of q.t. with no model in which all the predicates belong to $\sum_1 \cup \prod_1$ (answering another question of Mostowski's); thus $\sum_1^*$ represents the "lowest" level which contains "enough sets" so that it is always possible to find a model.

The penultimate section of this paper (§4) consists of some "enumeration theorems" (e.g., the potentially recursive functions are a recursively enumerable family of partial recursive functions), some of which are needed for the final section, and others of which are given as being of possible independent interest.

2. Characterization theorems.

Theorem 1. *P is a trial and error predicate if and only if* [illegible][5] [illegible]

[illegible] field [illegible]

[illegible]

The main result [illegible]

[illegible] Theorems 1 and 2, [illegible]

[illegible]

[illegible]

2. [illegible]

Theorem 1. [illegible]

$\underline{P} \in \Sigma_2 \cap \Pi_2$.

Proof:

Suppose P is a trial and error predicate. Then by the definition (cf. § 1), there is a g.r. function f such that for every $x_1, \ldots, x_m$:

$$P(x_1, \ldots, x_m) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_m, y) = 1$$

$$\overline{P}(x_1, \ldots, x_m) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_m, y) = 0$$

Now we observe that since f must approach either 0 or 1,

(1) $P(x_1, \ldots, x_m) \equiv \lim_{y \to \infty} f(x_1, \ldots, x_m, y) = 1$ implies that

(2) $P(x_1, \ldots, x_m) \equiv (y)(Ez)(f(x_1, \ldots, x_m, y) \neq 1 \Rightarrow (z > y \,\&\, f(x_1, \ldots, x_m, z) = 1))$

Thus $P \in \Pi_2$, and by (1) we have $P \in \Sigma_2$, since the predicate "$\lim_{y \to \infty} f(x_1, \ldots, x_m, y) = 1$" is in Σ_2.

To prove the other half of the theorem, assume

(3) $P(x_1, \ldots, x_m) \equiv (Ea)(b)R_1(x_1, \ldots, x_m, a, b)$

$\overline{P}(x_1, \ldots, x_m) \equiv (Ea)(b)R_2(x_1, \ldots, x_m, a, b)$

where R_1 and R_2 are recursive.

Let $T(x_1, \ldots, x_m, a, c)$ mean that a is the smallest integer such that $[(Ee)_{<c}$(e is the number of a proof (in, say, Robinson's arithmetic[9]) that $\overline{R}_2(x_1, \ldots, x_m, a, b)$ for some b) $\& \sim (Ee)_{<c}$ (e is the number of a proof that $\overline{R}_1(x_1, \ldots, x_m, a, b)$ for some b) .v. $(Ee)_{<c}$(e is the number of a proof that $\overline{R}_1(x_1, \ldots, x_m, a, b)$ for some b) $\& \sim (Ee)_{<c}$(e is the number of a proof that $\overline{R}_2(x_1, \ldots, x_m, a, b)$ for some b)].

2.2 Lemma

Proof:

Suppose F is a field and assume (definition) ... for the derivation (of ≤ 1), there is an r.e. predicate Q such that for every $x_1,\dots,x_n$:

$$P(x_1,\dots,x_n) \equiv \lim_{s\to\infty} Q(x_1,\dots,x_n,s) = 1$$

$$\bar{P}(x_1,\dots,x_n) \equiv \lim_{s\to\infty} Q(x_1,\dots,x_n,s) \neq 1$$

Now we observe that since P must be recursive ... ,

(2) $P(x_1,\dots,x_n) \equiv \lim_{s\to\infty} Q(x_1,\dots,x_n,s) = 1$ implies that

(3) $P(x_1,\dots,x_n) \equiv (\exists s)(\forall t)(Q(x_1,\dots,x_n,t) = 1$ or $(t < s)$ ... $Q(x_1,\dots,x_n,t) = 1)$

and $P \in \Sigma_2$, and so (by symmetry) $\bar{P} \in \Sigma_2$. Thus the predicate "$\lim_{s\to\infty} Q(x_1,\dots,x_n,s) = 1$" is in Σ_2.

To prove the other half of the theorem, assume

(4) $P(x_1,\dots,x_n) \equiv (\exists y)(\forall z) F_1(x_1,\dots,x_n,y,z)$

$\bar{P}(x_1,\dots,x_n) \equiv (\exists y)(\forall z) F_2(x_1,\dots,x_n,y,z)$

where F_1 and F_2 are recursive.

Let $Q(x_1,\dots,x_n,s)$ equal 1 if a is the smallest integer such that ... is the ... Robinson's arithmetic ... that $F_1(x_1,\dots,x_n,a,b)$ ... is the number of a proof that $F_2(x_1,\dots,x_n,a,b)$ ... $(\mu a)_{\omega}(a$ is the number of a proof of ... $F_1(x_1,\dots,x_n,a,b)$ and ... $(\mu a)_{\omega}(a$ is the number of a proof that $F_2(x_1,\dots,x_n,a)$ are both 0].

Define $\min_x P(x)$ as the least x such that P(x) if there is one, and as 0 otherwise. Further define:

(4) $g(x_1,\ldots,x_m,y) = \min_a T(x_1,\ldots,x_m,a,y)$

(5) $f(x_1,\ldots,x_m,y)=1\equiv T(x_1,\ldots,x_m,g(x_1,\ldots,x_m,y),y)$ & (Eb,c)(c < y & c is the number of a proof that $\overline{R}_2(x_1,\ldots,x_m, g(x_1,\ldots,x_m,y),b))$

(6) $f(x_1,\ldots,x_m,y) = 0 \equiv f(x_1,\ldots,x_m,y) \neq 1.$

g is general recursive, since from the definition of T we can determine whether or not there is an a such that $T(x_1,\ldots,x_m,a,y)$, once we are given $x_1,\ldots,x_m$ and y. There is at most one such a; hence $\min_a T(x_1,\ldots,x_m,a,y)$ equals this unique a if it exists and equals 0 otherwise. Moreover, if $P(x_1,\ldots,x_m)$ is true, then by (3) there is an a, and hence a least a, such that for every b, $R_1(x_1,\ldots,x_m,a,b)$; and by (3) there is no a such that for every b $R_2(x_1,\ldots,x_m,a,b)$. Hence, if $P(x_1,\ldots,x_m)$ is true and a is the least a such that for every b,$R_1(x_1,\ldots,x_m,a,b)$ holds, then for any sufficiently large y we will have $g(x_1,\ldots,x_m,y) = a$, and there will be an e < y such that e is the number of a proof that $\overline{R}_2(x_1,\ldots,x_m,a,b)$ for some b. (To verify this, we observe that since $P(x_1,\ldots,x_m)$ is assumed true, there is for every a' at least one b such that $\overline{R}_2(x_1,\ldots,x_m,a',b)$; and since a is "least", there is also for every a' < a at least one b such that $\overline{R}_1(x_1,\ldots,x_m,a',b)$. Assuming that we designate R_1 and R_2 by expressions in Robinson's arithmetic which strongly represent these predicates [so that each full sentence of these predicates is provable when true and refutable when false], there will thus

Determine $\min_y F(x)$ or to [illegible] $F(x)$ [illegible] is [illegible] following [illegible]:

(1) $S(x_1,\ldots,x_n,y) = \min_y F(x_1,\ldots,x_n,y)$

(2) $f(x_1,\ldots,x_n) = F(S(x_1,\ldots,x_n),x_1,\ldots,x_n)$ [illegible] (if $x > 0$ [illegible] $F_g(x_1,\ldots,x_n,y)$ [illegible])

(3) $A(x_1,\ldots,x_n,y) = \mu z\,(F(x_1,\ldots,x_n,z)) = 1$.

g is general recursive, [illegible] from the definition of g we can determine whether or not there is an n such that $F(x_1,\ldots,x_n,y)$ [illegible] $x_1,\ldots,x_n$ and y. [illegible] hence $\min_y F(x_1,\ldots,x_n,y)$ [illegible] if [illegible] and equals 0 otherwise. Moreover, if $F(x_1,\ldots,x_n)$ is true, then by (3) there is an n, [illegible] $F_g(x_1,\ldots,x_n,y,z)$; and by (3) there is [illegible] for every n [illegible] $(x_1,\ldots,x_n,y,z)$. [illegible] $F(x_1,\ldots,x_n)$ is false and [illegible] for every y, z, $(x_1,\ldots,x_n,y,z)$ holds, then for any sufficiently large y [illegible] $(x_1,\ldots,x_n,y) = 0$, and there will be no $z < y$ [illegible] of z and that $F_g(x_1,\ldots,x_n,y,z)$ [illegible]. [illegible] $F(x_1,\ldots,x_n)$ [illegible] the [illegible] $F_g(x_1,\ldots,x_n,y)$ [illegible] $F_g(x_1,\ldots,x_n,y,z)$ [illegible] by [illegible] functions [illegible] recursive [illegible] predicates [illegible].

be 2a provable propositions: $\overline{R}_1(x_1,\ldots,x_m,0,b_0)$, $\overline{R}_2(x_1,\ldots,x_m,0,b_0')$, $\ldots$, $\overline{R}_1(x_1,\ldots,x_m,a-1,b_{a-1})$, $\overline{R}_2(x_1,\ldots,x_m,a-1,b_{a-1}')$. Taking y to be sufficiently large so that each of these 2a propositions has at least one proof with number less than y, we see that $T(x_1,\ldots,x_m,a',y)$ does not hold for $a' < a$. And if y is also bigger than the least number of a proof that $\overline{R}_2(x_1,\ldots,x_m,a,b)$ for some b, then, checking the definition, we see that $T(x_1,\ldots,x_m,a,y)$ holds, and hence $g(x_1,\ldots,x_m,y) = a$.) Thus, if $P(x_1,\ldots,x_m)$ is true, for sufficiently large y we will have both $g(x_1,\ldots,x_m,y) = a$, where a is the smallest integer such that $(b)R_1(x_1,\ldots,x_m,a,b)$, and $f(x_1,\ldots,x_m,y) = 1$; and in a similar way we can show that if $P(x_1,\ldots,x_m)$ is false, then for sufficiently large values of y we will have both $g(x_1,\ldots,x_m,y) = a$, where a is the smallest integer such that $(b)R_2(x_1,\ldots,x_m,a,b)$, and $f(x_1,\ldots,x_m,y) = 0$. (f is clearly general recursive, in spite of the apparently unbounded existential quantifier (Eb), since its computation depends on examining only a finite number of proofs, namely those with number less than y.)- This completes the proof of the theorem.

Theorem 2. There exists a k such that P is a k-trial predicate if and only if P belongs to $\sum_1^*$, the smallest class containing the recursively enumerable predicates and closed under truth-functions.

Proof : Suppose P is a k-trial predicate. Then by the definition (cf. § 1) there is a g.r. function f such that

$$(1)\quad P(x_1,\ldots,x_m) \equiv \lim_{y\to\infty} f(x_1,\ldots,x_m,y) = 1$$

(2) there are <u>at most</u> k integers y, for each $x_1, \ldots, x_m$, such that $f(x_1, \ldots, x_m, y) \neq f(x_1, \ldots, x_m, y+1)$.

Now define $Y_i(x_1, \ldots, x_m)$ (for $i = 1,2,\ldots,k$) as meaning that there are at least i integers y such that $f(x_1, \ldots, x_m, y) \neq f(x_1, \ldots, x_m, y+1) \,\&\, f(x_1, \ldots, x_m, a_i+1) = 1$, where a_i is the ith integer y, in order of magnitude, such that $f(x_1, \ldots, x_m, y) \neq f(x_1, \ldots, x_m, y+1)$; and define $N_i(x_1, \ldots, x_m)$ as meaning that there are at least i integers y such that $f(x_1, \ldots, x_m, y) \neq f(x_1, \ldots, x_m, y+1) \,\&\, f(x_1, \ldots, x_m, a_i+1) \neq 1$. Finally, define $Y_0(x_1, \ldots, x_m)$ as meaning that $f(x_1, \ldots, x_m, 0) = 1$, and $N_0(x_1, \ldots, x_m)$ as meaning that $f(x_1, \ldots, x_m, 0) \neq 1$. Then all the predicates Y_i and N_i are recursively enumerable, and we have:

$$\begin{aligned} P(x_1, \ldots, x_m) \equiv\ & Y_k(x_1, \ldots, x_m) \vee (Y_{k-1}(x_1, \ldots, x_m) \,\&\, \overline{N}_k(x_1, \ldots, x_m)) \\ & \vee (Y_{k-2}(x_1, \ldots, x_m) \,\&\, \overline{N}_{k-1}(x_1, \ldots, x_m)) \vee \ldots \\ & \vee (Y_0(x_1, \ldots, x_m) \,\&\, \overline{N}_1(x_1, \ldots, x_m)). \end{aligned}$$

In proving the other half of the theorem, we will confine attention to one-place predicates (or sets), since the n-place case introduces no additional ideas. Let $P \in \sum_1^*$. Then

$P = (A_1 - B_1) \cup (A_2 - B_2) \cup \ldots \cup (A_n - B_n)$ for some n where the A_i and B_i are r.e. [noting that every r.e. predicate has the form A-B, e.g., by taking $B = \Lambda$; every complement of an r.e. predicate has the form A-B, e.g., taking $A = \Lambda$; and the predicates of this form are closed under intersection, since $(A-B)(C-D) = AC - (B \cup D)$. But by the familiar disjunctive normal form, every truth-function of r.e. predicates can be written as a disjunction whose terms are just intersections of r.e. predicates and their complements; hence, a disjunction of the kind

(2) there are at most n integers [illegible] such that $f(x_1,\ldots,x_n) \neq f(x_1,\ldots,x_n,$ [illegible]$)$.

We define $X_i(x_1,\ldots,x_n)$ (for $i = 1,\ldots,n$) as meaning that there are at least i integers y such that $f(x_1,\ldots,x_n,y) \neq f(x_1,\ldots,x_n,y+1)$ [illegible] integer y, in order of magnitude, [illegible] and define [illegible] as meaning that there are at least [illegible] integers y such that [illegible]. [illegible] recursively enumerable. [illegible] we have

[illegible]

In proving the other half of the theorem [illegible]

[illegible]

given.] Following Kleene [2], let $T(e,x,y)$ mean that y is the number of a proof (or computation) that the number x belongs to the r.e. set with gödel number e (or the domain of the partial recursive function with number e, in Kleene's formalism). We define $f(x,y)$ as follows (where $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are gödel numbers of $A_1,\ldots,A_n$ and $B_1,\ldots,B_n$):

$$f(x,y) = 1 \quad \text{if there are } i \le n,\ e < y \text{ such that } (T(a_i,x,e)\ \&\ (e)_{<y}\ \overline{T}(b_i,x,e))$$
$$f(x,y) = 0 \quad \text{otherwise}$$

Then f has the properties (1) and (2) (taking $k = 2n$).

To verify this, first assume $P(x)$ holds. Then $x \varepsilon A_i - B_i$, for some $i \le n$. Then for some e, $T(a_i,x,e)$ (by the Normal Form theorem and the fact that a_i is a gödel number of A_i); and for no e' is it the case that $T(b_i,x,e')$. Thus $f(x,y) = 1$ whenever $y > e$. On the other hand, if $P(x)$ does not hold, then $x \varepsilon B_i$ for every i such that $x \varepsilon A_i$. Let N be any integer larger than all of $e_1,\ldots,e_n$, where e_i is the smallest number which is a gödel number of a proof that $x \varepsilon B_i$, if there is such a proof, and $e_i = 0$ otherwise. Then if $y > N$ and there is an $e < y$ such that $T(a_i,x,e)$, there is also an $e < y$ such that $T(b_i,x,e)$; so in this case $f(x,y) = 0$ whenever $y > N$. Thus we have verified property (1), or

$$P(x) \equiv \lim_{y \to \infty} f(x,y) = 1.$$

To verify property (2), suppose $f(x,y) \neq f(x,y+1)$. There are two cases:

case (a) $f(x,y) = 1$, $f(x,y+1) = 0$.

In this case, by the definition of f, there are $i \leq n$, $e < y$ (and hence $< y + 1$) such that $T(a_i,x,e) \,\&\, (e)_{<y} \overline{T}(b_i,x,e)$; but $f(x,y + 1) = 0$, so there must be an $e' < y + 1$ such that $T(b_i,x,e')$. Hence we must have $e' = y$, and e' must $= e_i$ (the smallest number of a proof that $x \varepsilon B_i$). Since there are only n sets B_i altogether, and there is only one e_i for each B_i, this case can arise for at most n values of y.

case (b) $f(x,y) = 0$, $f(x,y + 1) = 1$

In this case, by the definition of f, there are $i \leq n$, $e<y+1$ such that $T(a_i,x,e) \,\&\, (e) < y+1 \ \overline{T}(b_i,x,e)$; but $f(x,y) = 0$, so e cannot be $< y$. Hence we must have $e = y$, and e must be the smallest number of a proof that $x \varepsilon A_i$. Since there are only n sets A_i altogether, this case can arise for at most n values of y.

Combining the two cases, we see that $f(x,y) \neq f(x,y+1)$ can hold for at most 2n values of y.---This completes the proof.

3. Applications to metalogic.

Lemma 1. Let A be a well formed formula of quantification theory containing only one predicate letter, say, P. Let A be true when P is interpreted as standing for F, where F is some predicate of non-negative integers, and the variables are interpreted as ranging over non-negative integers. Let R be a 1-1 mapping of the non-negative integers into the family of all sets of non-negative integers, and G a predicate of non-negative integers such that:

1) $a \neq b \Rightarrow R(a) \cap R(b) = \Lambda$

2) $\bigcup_a R(a) = N$ (the set of all non-negative integers)

3) $R(a) \neq \Lambda$

4) $G(x_1,\ldots,x_m) \equiv (Ey_1,\ldots,y_m)(F(y_1,\ldots,y_m) \& x_1 \varepsilon R(y_1) \& \ldots \& x_m \varepsilon R(y_m))$

<u>Then A is also true when P is interpreted as standing for the predicate G.</u>

<u>Proof</u> : The mapping

$$n \longleftrightarrow \text{any member of } R(n)$$

is a one-many mapping of N <u>onto</u> N under which $F \longleftrightarrow G$ (i.e.,[10] $F = T^{-1}(G)$, where T is the above mapping). Since the pair[11] $< N,F >$ is a model of A, so must the pair $< N,G >$ be. q.e.d. (Since T is not necessarily one-one, this Lemma is false for predicate calculus with identity.)

<u>Theorem</u> 3. <u>Every consistent formula of quantification theory has a model in</u> $\sum_1^*$.

<u>Proof</u> : If A contains m predicate letters P_i, each of which is at most n-place, we construct an A' which is obviously satisfiable if and only if A is, and which has a single n + 1-place predicate letter and m distinct individual constants by replacing $P_i(x_1,\ldots,x_r)$ $(1 \leq r \leq n)$ by $\underbrace{P(x_1,\ldots,x_r,a_i,\ldots,a_i)}_{n+1 \text{ argument places}}$.

Suppose A' has a model in $\sum_1^*$. Then the predicates P_i defined as follows:

$$P_1(x_1,\ldots,x_{r_1}) =_{df} P(x_1,\ldots,x_{r_1},a_1,\ldots,a_1)$$

$$\vdots$$

$$P_m(x_1,\ldots,x_{r_m}) =_{df} P(x_1,\ldots,x_{r_m},a_m,\ldots,a_m)$$

are also in Σ_1^*. Hence it suffices to show that if A' is consistent then A' has a model in Σ_1^* ; it will thon automatically follow that if A is consistent then A has a model in Σ_1^*.

Finally, A' has a model in Σ_1^* if and only if its existential quantification with respect to $a_1,\dots,a_m$ has a model in Σ_1^*. Hence the theorem reduces to the following lemma:

Lemma 2 : Every consistent formula of quantification theory with one predicate letter and no individual constants has a model in Σ_1^*.

To prove this we start[12] with a model in $\Sigma_2 \cap \Pi_2$ and modify it so as to obtain a model in Σ_1^*. Accordingly, let P be the sole predicate letter in A, and let A be true when P is interpreted as standing for the predicate F, where $F \in \Sigma_2 \cap \Pi_2$. By Theorem 1, there is a general recursive function $f(x_1,\dots,x_n,y)$ such that (for all $x_1,\dots,x_n$)

$$F(x_1,\dots,x_n) \equiv \lim_{y \to \infty} f(x_1,\dots,x_n,y) = 1$$

$$\overline{F}(x_1,\dots,x_n) \equiv \lim_{y \to \infty} f(x_1,\dots,x_n,y) = 0$$

We define sets of integers R(i) as follows[13]:

$$\text{if } i \neq 0,\ R(i) = \{J(b,i)\}$$

where b is the smallest integer such that (for all $x_1,\dots,x_n$)

$$y > b \ \& \ x_1,\dots,x_n \leq i \Rightarrow f(x_1,\dots,x_n,y) = f(x_1,\dots,x_n,b)$$

(i.e., b is a "modulus of convergence" of f for $x_1,\dots,x_n \leq i$).

We take R(0) as the set of all integers not belonging to any set R(i), $i \neq 0$.

It is easily proved that the sets R(i) are all disjoint and non-empty. (Towards disjointness, use the fact that J(a,b) =

and also in $\Sigma_n^{(1)}$. Hence it suffices to show that if it is pos-
sible that it is a member of $\Sigma_n^{(1)}$ it will then constructively
follow that it is constructive that it may be found in $\Sigma_n^{(1)}$.

Finally, it was a result in $\Sigma_n^{(1)}$ if and only if its existential quantification with respect to $b_1, \ldots, b_n$ has a value in $\Sigma_n^{(1)}$. Hence the theorem reduces to the following lemma:

Lemma 3. — Every existential formula of quantification-free [illegible] with one variable [illegible] in $\Sigma_n^{(1)}$.

To prove this we [illegible] that a member of $\Sigma_n^{(1)}$ [illegible] T_n [illegible] to obtain a value in $\Sigma_n^{(1)}$. [illegible] of the form [illegible] $(T_1, \ldots)$ [illegible] T_n, where $\in \Sigma_n^{(1)}$, [illegible] $f(b_1, \ldots, b_n)$ [illegible] $(b_1, \ldots, b_n)$

$$f(b_1, \ldots, b_n) = \ldots f(b_1, \ldots, b_n) = 1$$

$$f(b_1, \ldots, b_n) = \ldots f(b_1, \ldots, b_n) = 0$$

[illegible]

$$\{f(b_1, \ldots) = 1\}$$

[illegible] $b_1, \ldots, b_n$ [illegible]

$f(b_1, \ldots, b_n) = \ldots f(b_1, \ldots, b_n)$ [illegible] $f(b_1, \ldots, b_n)$

[illegible] $(b_1, \ldots, b_n) = 1$.

[illegible]

[illegible]

[illegible]

$= J(c,d)$ implies that $a = c$ and $b = d$; and towards non-emptiness observe that for any k, $J(k,0) \varepsilon R(0)$.) And by the definition of R(0) every integer is in one of the sets R(i).

Now we define a predicate G as follows:

$$G(x_1,\ldots,x_n) \equiv (Ey_1,\ldots,y_n)(F(y_1,\ldots,y_n) \& x_1 \varepsilon R(y_1) \& \ldots$$
$$\ldots \& x_n \varepsilon R(y_n))$$

By Lemma 1, A is true when P is interpreted as standing for G. Hence it only remains to prove that $G \varepsilon \sum_1^*$.

(For simplicity, sequences "$x_1,\ldots,x_n$", "$y_1,\ldots,y_n$", etc., will henceforth be abbreviated by capital letters. E.g., in this notation the definition of G would be written:

$G(X) \equiv (EY)(F(Y) \& X \varepsilon R(Y))$.)

To prove that $G \varepsilon \sum_1^*$, observe that for any integer x, there are only two possibilities: $x \varepsilon R(0)$, and $x \varepsilon R(L(x))$. Hence for any n integers $x_1,\ldots,x_n$ there are just 2^n possible cases:

1) $x_1,\ldots,x_n \varepsilon R(0)$

2) $x_1,\ldots,x_{n-1} \varepsilon R(0), x_n \varepsilon R(L(x_n))$

.
.
.

2^n) $x_1 \varepsilon R(L(x_1)),\ldots,x_n \varepsilon R(L(x_n))$

Moreover, the truth value of G(X) on the assumption that any given case holds can be <u>effectively</u> determined: For instance, the truth value of G(X) on the assumption that case 1) holds is that of $F(0,0,\ldots,0)$ (which we will assume given); while if, say, case 2^n-1) holds, the truth value of G(X) is that of $F(L(x_1),\ldots,L(x_{n-1}),0)$. In this case, we simply find the <u>largest</u>

of the numbers $L(x_1),\ldots,L(x_{n-1})$. Suppose it is $L(x_j)$. Then $G(X)$ is true if $f(L(x_1),\ldots,L(x_{n-1}),0,K(x_j)) = 1$, and false otherwise. And similarly with all the other cases.

We can now write down a series of zeros and ones which will terminate in 1 if $G(X)$ is true and in 0 if $G(X)$ is false, as follows:

<u>We assume</u> first that $x_j \varepsilon R(L(x_j))$, where $L(x_j)$ is the largest of the numbers $L(x_i)$. Then we compute the truth-value of $G(X)$ according to the assumption, and put down 1 as our "first trial answer" if the value is "truth" and 0 if the value is "falsity". The first trial answer is never revised unless an integer k is generated such that $K(x_j) < k$, but for some $Z \leq L(x_j)$, it is not the case that $f(Z,K(x_j)) = f(Z,k)$. If this ever happens, then $f(Z,y)$ is <u>not</u> equal to $f(Z,K(x_j))$ for all $y > K(x_j)$, $Z \leq L(x_j)$, and $x_j \varepsilon R(0)$.

If we ever discover that $x_j \varepsilon R(0)$, then we pick the largest of the remaining numbers $L(x_i)$ and rpeat the whole reasoning to arrive at our next trial answer. (If $L(x_j')$ is the largest of the remaining numbers, we can determine the truth-value of $G(X)$ on the assumption that $x_{j'} \varepsilon R(L(x_{j'}))$, because we now know that $x_j \varepsilon R(0)$, and so it suffices to know the truth-value of $F(Z)$ for $Z \leq L(x_{j'})$ to compute that of $G(X)$.)

In this way we cannot change our trial answer more than n times (since, except for the trial answer corresponding to case 1), a trial answer is put down only when it is assumed that $x_i \varepsilon R(L(x_i))$ for some i; and such an assumption is either retained forever in our procedure---in which case it is correct---or

abandoned at some time and never subsequently reinstated.

Let the above procedure for putting down trial answers be mechanized, and program the Turing machine so that at any stage y it repeats the last number it put down, if no new trial answer is forthcoming at that stage. Let f(X,y) = the number put down by the machine at the yth stage. Then f satisfies the conditions listed in Theorem 2, and it follows that $G \in \sum_1^*$. q.e.d.

Hitherto we have considered models in which the domain (the range of the individual variables) was the set of all non-negative integers. For models of this kind, Theorem 3 is false for predicate calculus with identity, since there are even consistent formulas with no infinite model at all. If we generalize slightly, by allowing the domain to be any recursive set, then the question whether Theorem 3 extends also to predicate calculus with identity remains open. We are, however, able to prove:

<u>Theorem 4</u>. <u>Every consistent formula of predicate calculus with identity has a recursive model with a Π_1 domain</u>.

<u>Proof</u>: In the foregoing proof, it suffices to modify the definition of R(0) by taking $R(0) = \{J(b,0)\}$, where b is the smallest integer such that for all y > b, f(0,...,0,y) = f(0,...,0,b).

Let $S = \{J(b,i) \mid (y)_{>b}(Z)(Z \leq i \Rightarrow f(Z,y) = f(Z,b)) \,\&\, (b')_{<b} (EZ)(Ey)_{>b'} (Z \leq i \,\&\, f(Z,y) \neq f(Z,b')).\}$ Then we now have $S = \bigcup_i R(i)$. Since the mapping

$$n \longleftrightarrow \text{any member of } R(n)$$

is now one-one (because R(n) is now a unit set, for all n), the argument of Lemma 1 shows that < S,G > is a model for A, where

G is defined as in the preceding proof. Define G^* as follows: $G^*(X)$ is true if the truth-value of $G(X)$ is "truth" <u>on the assumption that case 2^n) holds</u> (we showed above that this could be effectively determined), and $G^*(X)$ is false if the truth-value of $G(X)$ is "falsity" on the assumption that case 2^n) holds. Then G^* is a recursive predicate (we make free use of Church's Thesis; however, it is straightforward to eliminate it by the techniques of [2]) and G^* agrees with G whenever case 2^n holds: hence, whenever all the arguments ε S. Thus $< S, G^* >$ is also a model for A.

It remains only to show that S is a Π_1 set (i.e., S has a recursively enumerable complement). To do this, we observe that the unbounded existential quantifier (Ey) can be eliminated by using the alternative definition:

$$S = \{ J(b,i) \mid (y)_{>b}(Z)(Z \leq i \Rightarrow f(Z,y) = f(Z,b)) \ \& \ (b')_{<b}((Z)(Z \leq i \Rightarrow f(Z,b') = f(Z,b)) \Rightarrow (Eb'')_{<b}(EZ)(Z \leq i \ \& \ f(Z,b') \neq f(Z,b'') \ \& \ b' < b'')) \}.$$

(To verify this, note that if b is not the smallest modulus of convergence of f, for arguments bounded by i, then either b is not a modulus of convergence at all, or there is a b' smaller than b such that $f(Z,b') = f(Z,b'') = f(Z,b)$ for all b'' with $b' < b'' \leq b$, and all $Z \leq i$.)

4. <u>Some enumeration theorems</u>. Two of the following theorems will be used in the last section. The others are given because of their possible independent interest.

Terminology: $q_1^n, q_2^n, \ldots$ will be the n-place partial recursive functions, in any standard enumeration. $W_1^n, W_2^n, \ldots$ will be the n-place r.e. predicates in the standard enumeration (i.e., $W_i^n = \{ X \mid (Ey)T^n(i,X,y) \}$, where T^n is the n+2-place predicate "y is the number of a proof (or computation) that the n-tuplet X satisfies the n-place r.e. predicate with number i", as in [2]). A family F of n-place partial recursive functions will be called a recursively enumerable family of partial recursive functions if there is an n+1-place partial recursive function h, which "enumerates" the family---i.e., such that for every f ε F there is an i such that[14] $f = \lambda_X h(i,X)$ and conversely, $\lambda_X h(i,X) \varepsilon F$ for every i. (Thus $F = \{ \lambda_X h(0,X), \lambda_X h(1,X), \ldots \}$.) A family F of n-place recursively enumerable predicates will be called a recursively enumerable family, if there is an n+1-place r.e. predicate R(i,X) such that for every R ε F there is an i such that $P = \{ X \mid R(i,X) \}$ and conversely, $\{ X \mid R(i,X) \}$ ε F for every i. Similarly, a family F of pairs $< P,Q >$ of n-place recursively enumerable predicates is called a recursively enumerable family if there are n+1-place recursively enumerable predicates R_1, R_2 such that for every $< P,Q >$ ε F there is an i such that $P = \{ X \mid R_1(i,X) \}$ and simultaneously $Q = \{ X \mid R_2(i,X) \}$, while conversely $< \{ X \mid R_1(i,X) \}, \{ X \mid R_2(i,X) \} >$ ε F for all i.

Henceforth, Q^n will be the family of all n-place partial recursive functions, G^n the family of all n-place general recursive functions, F^n the family of all "finite" functions (here: functions whose domain consists of the first m n-tuplets, in the standard lexicographic enumeration, for some m; or the integers

< m, for some m, in the case of singulary functions), R^n the family of all n-place partial recursive functions with recursive domain, and P^n the family of all n-place potentially recursive functions (partial recursive functions which agree where defined with some general recursive function--- it is known that there exist members of Q^n which are not in P^n.) It is well known that G^n is not a recursively enumerable family. However, I shall prove:

Theorem 5: The family $G^n \cup F^n$ is a recursively enumerable family, for all n.

Theorem 6: The family R^n is a recursively enumerable family, for all n.

Theorem 7: The family P^n is a recursively enumerable family, for all n.

Theorem 8: The family of all pairs $< W_i^n, W_j^n >$ such that $W_i^n \cap W_j^n = \Lambda$ is a recursively enumerable family, for every n.

Proof of Theorem 5. To simplify our notations, we give the following proofs for the case of singulary functions and predicates. The proofs for the general case may be obtained by inserting superscript "n" (we omit the superscript "1" for the singulary case) and putting "X" for "x".

We define q_i^*, for each i, as follows:

(i) $q_i^*(x) = 2^a 3^b$ if $a = q_i(x)$ and b is the smallest integer which is the number of a proof (in, say, Robinson's arithmetic) that $(y)_{<x} q_i(y)$ is defined.

(ii) $q_i^*(x)$ is not defined if $q_i(y)$ is not defined for any $y \leq x$

(In the general case, $Y < X$ has to be interpreted as meaning that the n-tuplet Y precedes the n-tuplet X in the lexicographic ordering.)

We also define $\psi(i,x) = y \equiv (Ea,b)(y = 2^a 3^b \& \phi_1(i,x) = a \& b$ is the smallest integer which is the number of a proof that $q_i(n)$ is defined for all $n < x$), where $\phi_1(i,x)$ (in the general case, $\phi_n(i,X)$) is the partial recursive function introduced in [2] with the property that for all i,x $q_i(x) = \phi_1(i,x)$; and we see that the predicate $\psi(i,x) = y$ is r.e. (since only existential quantification, conjunction, and r.e. predicates occur in the definition) which implies that the function $\psi(i,x)$ is partial recursive; and it is obvious (comparing the definitions) that for all i,x

$$q_i^*(x) = \psi(i,x)$$

---so that the functions q_0^*, q_1^*,... form a recursively enumerable family. Moreover, from (i) we have[15] that $q_i^*(x)$ is defined => $q_i(y)$ is defined for all $y \leq x$ => $q_i^*(y)$ is defined for all $y \leq x$. Thus $q_i^* \varepsilon G \cup F$.

When $x = 2^a 3^b$, let $(x)_0 = a$ (as in [2]). Then, since $q_i^*(x)$ is of the form $2^a 3^b$ when defined, $q_i^* \varepsilon G \cup F$ => $(q_i^*)_0 \varepsilon G \cup F$. And if $q_i \varepsilon G \cup F$, it is easily verified that $q_i = (q_i^*)_0$ (I.e., $\lambda_x q_i(x) = \lambda_x(q_i^*(x))_0$). Thus $(q_0^*)_0, (q_1^*)_0, \ldots$ is an enumeration in some order of all and only the functions in $G \cup F$. But $(q_i^*)_0 = \lambda_x(q_i^*(x))_0 = \lambda_x(\psi(i,x))_0$, and $(\psi(i,x))_0$ is evidently partial recursive (regarded as a function of both i and x) since ψ is and $\lambda_x(x)_0$ is.- This completes the proof of Theorem 5.

<u>Corollary</u>. <u>The n-place recursive predicates are a recursively enumerable family, for every n</u>. (This was first proved by Dekker).

<u>Proof</u>. It suffices to take the predicate $(\psi^n(i,X))_0 = 1$, where ψ is the function constructed in the preceding proof (in the general case we add a superscript n, since the function ψ depends on the number n of argument-places considered). For, if P is recursive, then its characteristic function[16] is general recursive, and so $P = \{ X | (\psi^n(i,X))_0 = 1 \}$ for some i, by the preceding proof. Conversely, if $\lambda_X(\psi^n(i,X))_0$ is general recursive, then $\{ X | (\psi^n(i,X))_0 = 1 \}$ is recursive, and if $\lambda_X(\psi^n(i,X))_0$ is "finite", then $\{ X | (\psi^n(i,X))_0 = 1 \}$ is finite, and hence recursive. -This completes the proof.

<u>Proof of Theorem 6</u>. Let t be a g.r. function such that $W_{t(0)}$, $W_{t(1)}, \ldots$ are the recursive sets (n-place predicates, in the general case) in some order. (By the preceding Corollary these form a recursively enumerable family; so by the Iteration Theorem[17] such a function exists). Define:

$$\xi(i,x) = y \equiv (\psi(K(i),x))_0 = y \,\&\, x \in W_{t(L(i))},$$

so that $\xi(i,x)$ is partial recursive, and $\lambda_x \xi(i,x)$ agrees with $(q^*_{K(i)})_0$ on numbers in $W_{t(L(i))}$ and is undefined on numbers outside of $W_{t(L(i))}$. Then the domain of $\lambda_x \xi(i,x)$ is $W_{t(L(i))}$ if $(q^*_{K(i)})_0 \in G$, and is finite if $(q^*_{K(i)})_0 \in F$. So, in either case the domain of $\lambda_x \xi(i,x)$ is recursive, and $\lambda_x \xi(i,x) \in R$. On the other hand, if $q_i \in R$, then there is a j such that the domain of q_i is $W_{t(j)}$. Also, the function f defined by:

$f(x) = q_i(x)$ if $x \in W_{t(j)}$ and $f(x) = 0$ otherwise is general

recursive (because $W_{t(j)}$ is recursive). Then $\lambda_x q_i(x)$ = the function which agrees with f on numbers in W_j and is undefined outside of $W_j = \lambda_x \xi(J(e,j),x)$, where e is any gödel number of f (recalling that $(q_e^*)_0 = q_e$ when $q_e \varepsilon G$). Thus $\lambda_x \xi(0,x)$, $\lambda_x \xi(1,x)$,... is an enumeration in some order of the members of R.

Proof of Theorem 7. Define:

$$H(i,x) = y \equiv (\psi(K(i),x))_0 = y \ \& \ x \varepsilon W_{L(i)},$$

so that H(i,x) is partial recursive and $\lambda_x H(i,x)$ agrees with $(q^*_{K(i)})_0$ on numbers in $W_{L(i)}$, and is undefined on numbers outside of $W_{L(i)}$. Then $\lambda_x H(i,x)$ agrees with a recursive function where defined (noting that every function in F can be extended to a general recursive function, e.g., by giving the value 0 wherever the function was not defined), and has as its domain either $W_{L(i)}$ or some finite set: so, in either case, an r.e. domain. Thus $\lambda_x H(i,x)$ is potentially recursive, for all i. On the other hand, if $q_i \varepsilon$ P, then q_i agrees with some general recursive q_e where defined, and has some r.e. set W_j as its domain. Hence $q_i = \lambda_x H(i,x)$, where $i = J(e,j)$.

Proof of Theorem 8. It suffices to take $R_1(i,x) \equiv (Ey)(T(K(i),x,y) \ \& \ (z)_{\leq y} \overline{T}(L(i),x,z))$ and $R_2(i,x) \equiv (Ey)(T(L(i),x,y) \ \& \ (z)_{\leq y} \overline{T}(K(i),x,z))$. Then R_1, R_2 are defined from r.e. predicates using conjunction, existential quantification, and bounded universal quantification, and are therefore r.e.. Also, it is easily seen that $R_1 \Rightarrow \sim R_2$: so

$\{x | R_1(i,x)\} \cap \{x | R_2(i,x)\} = \Lambda$ for all i. Finally, if $W_i \cap W_j = \Lambda$, then $\{x | R_1(e,x)\} = W_1$ and $\{x | R_2(e,x)\} = W_2$,

where e = J(i, j); so $< W_i, W_j > = < \{x \mid R_1(e,x)\}, \{x \mid R_2(e,x)\} >$. -This completes the proof.

I employed the construction used to prove Theorem 8 in [5] in the course of giving an example of an axiomatizable theory with only monadic predicates (all of whose finitely axiomatizable subtheories were accordingly decidable by elimination of quantifiers) in which any two disjoint r.e. sets were exactly separable (and which was, therefore, essentially undecidable), but without explicitly stating the theorem.

5. A "hierarchy" theorem; moduli of convergence; moduli of oscillation.

So far we have looked only at the union of the k-trial predicates for all k. It is, however, also natural to ask whether they form a "sub-hierarchy"; that is, whether for each k, there is a predicate which is a k + 1-trial predicate but not a k-trial predicate; and, if so, how this "sub-hierarchy" is related to the larger class of "trial and error" predicates, ---i.e., to $\sum_2 \cap \prod_2$. In this section, we discuss these questions.

Lemma 3. The class of k-trial predicates (for each k) is closed under negation.

Proof: Let P be a k-trial predicate and let f_P be the corresponding function such that

(1) $P(X) = \lim_{y \to \infty} f_P(X,y) = 1$

(2) There are at most k values of y such that $f(X,y) \neq f(X,y+1)$ for each X

Define:

$$f_P'(X,y) = 0 \quad \text{if } f_P(X,y) = 1$$

$$f_P'(X,y) = 1 \quad \text{otherwise}$$

Then it is easily verified that

(1') $\overline{P}(X) \equiv \lim_{y\to\infty} f_P'(X,y) = 1$ (recalling that $\lim_{y\to\infty} f_P(X,y)$ always exists)

(2') There are at most k numbers y such that $f_P'(X,y) \neq f_P'(X,y+1)$ for each X.

Thus $\overline{P}$ is a k-trial predicate if P is. q.e.d.

Theorem 9. (Enumeration Theorem for k-trial predicates:) The n-place k-trial predicates are enumerated by a single n+1-place k+1-trial predicate: i.e., there is a k+1-trial predicate P(i,X) such that for every n-place k-trial predicate T there is an e such that (for all X) T(X) ≡ P(e,X), and such that, conversely, P(e,X) is a k-trial predicate for every value of e.

Proof: (As before, we use the notations for the singulary case:) We saw in the proof of Theorem 2 that every k-trial predicate can be written in the "normal form"

(A) $Y_k \cup (Y_{k-1} - N_k) \cup \ldots \cup (Y_0 - N_1)$

where

(i) Y_0 is recursive

(ii) Y_i and N_i are r.e. (for all i)

(iii) $Y_i \cap N_i = \Lambda$ (for all i)

Conversely, any predicate of the form A is k-trial; for given a predicate expressed in the form (A) one can program a

Turing Machine as follows:

The machine gives "yes" (or "1") as its first "trial answer" if $x \in Y_0$ and "no" (or "0") otherwise (since Y_0 is recursive the machine can be programmed to do this). At each subsequent stage $\sum$ in its computation, the machine puts down "yes" as its $\sum$ + 1st trial answer if there are $e < \sum$, $i < k$ such that $T(y_i,x,e)$ & $(e')_{<\sum}(\overline{T}(n_{i+1},x,e'))$, or if $x \in Y_0$ and there is no $e < \sum$ such that $T(n_1,x,e)$, or if $T(y_k,x,e)$ for some $e < \sum$, where $y_1,y_2,\ldots,y_k$ and $n_1,n_2,\ldots,n_k$ are gödel numbers of $Y_1,Y_2,\ldots,Y_k$ and $N_1,N_2,\ldots,N_k$ respectively; and otherwise the $\sum$ + 1st trial answer is "no". By the argument used to prove the second half of Theorem 2, this series of "yesses" and "nos" converges to "yes" if there is a Y_i such that $x \in (Y_i - N_{i+1})$ or if $x \in Y_k$; and the machine will not "change its mind" more than 2k times. But in fact, the machine will not "change its mind" more than k times. For the machine "changes its mind" only when:

case (a) x is generated in one of the N_i, $i > 0$, and Y_{i-1} was the only Y_i in which x had been previously generated without having also been generated in N_i; or

case (b) x is generated in one of the Y_i, $i > 0$, and x has not already been generated in N_{i+1} (and Y_i is the only Y_i in which x has been generated without having been previously generated in N_{i+1}; and x has already been generated in N_1 in case x belongs to Y_0).

But for each value of $i > 0$, only one of these cases can arise by the disjointness of Y_i and N_i. Thus any predicate of

the form (A) is a k-trial predicate. Note that the proof uses both the recursiveness of Y_0 and the disjointness of Y_i, N_i. Of course, the reference to Turing Machines is inessential: What we have given is a definition of a g.r. function (as in the proof of Theorem 2) with the properties needed to show that (A) is a k-trial predicate.

Now, suppose we recursively enumerate all the predicates of the form (A) (using for the purpose our recursive enumeration of the recursive sets (Corollary to Theorem 5) and of the pairs N_i, Y_i of disjoint r.e. sets (Theorem 8). The above argument does <u>not</u> provide a <u>uniform</u> way of programming a Turing Machine to generate the appropriate sequences of "yesses" and "nos"--- simply because the argument did not use only the recursiveness of Y_0, but assumed that the decision method for Y_0 was available, and there is no effective procedure for going from a gödel number of a recursive set W_i to a decision method. However, we can <u>effectively</u> go from an expression of the form (A) (assuming we are given the gödel numbers of the Y_i and N_i), to a program which expresses (A) as a k+1-trial predicate. Namely, since the only thing we lack is a decision procedure for Y_0, we define the "first trial answer" to always be "no", and the $\sum$ + 1st trial answer as above, except that Y_0 is now treated exactly like the other Y_i, i.e., the $\sum$ + 1st trial answer is "yes" if there is an $e < \sum$ such that $T(y_i,x,e) \;\&\; (e')_{<\sum} (\overline{T}(n_{i+1},x,e'))$, where now $0 \leq i \leq k$ (with the second factor omitted for $i = k$). The machine can now "change its mind" in one more way: namely, there will be a "change of mind" corresponding to $i = 0$ if x is

generated in Y_0 before it is generated in N_1, and the currently accepted answer is "no". Previously, if $x \varepsilon Y_0$, "yes" was taken as the first trial answer (using the decision method for Y_0) which is why there was no "change of mind" corresponding to $i = 0$. Thus the predicates of the form (A) are individually k-trial predicates, but the best result that we can get "with uniformity"--- i.e., if we want a "meta-program" from going from a normal form to a program for writing down the corresponding sequences of yesses and nos--- is that they are k+1-trial predicates. That this is indeed "best possible" follows from the argument of Theorem 10, below, which shows that any predicate that enumerates all the k-trial predicates (of a given number of argument-places) cannot itself be k-trial.

From these facts, we can easily obtain our theorem. Let t be a g.r. function which enumerates the recursive sets (i.e., such that $W_{t(0)}, W_{t(1)}, \ldots$ are all the recursive sets, in some order) and let ℓ, m be g.r. functions such that $< W_{\ell(0)}, W_{m(0)} >$, $< W_{\ell(1)}, W_{m(1)} >, \ldots$ are all the pairs of disjoint r.e. sets in some order (the existence of such functions---even primitive recursive functions with these properties---follows from our Theorem 5, Corollary, and our Theorem 8 by the Iteration Theorem (cf. n. 17)). Let $K_0, K_1, \ldots, K_k$ be recursive "k+1-tupling functions" ---recursive functions such that the k+1-tuplets $< K_0(0), \ldots, K_k(0) >$, $< K_0(1), \ldots, K_k(1) >, \ldots$ are all the k+1-tuplets of non-negative integers in some order. Then

$$P(i,x) =_{df} x \varepsilon [W_{\ell(K_k(i))} \cup (W_{\ell(K_{k-1}(i))} - W_{m(K_k(i))}) \cup \ldots \cup (W_{\ell(K_1(i))} - W_{m(K_2(i))}) \cup (W_{t(K_0(i))} - W_{m(K_1(i))})]$$

enumerates all the predicates of the form (A), and so all k-trial predicates. That the predicate P(i,x) is k+1-trial (regarded as a predicate of both i and x) follows from the preceding argument.

Corollary. Not every k-trial predicate corresponds to a function f which is primitive recursive; but all k-trial predicates P correspond as k+1-trial predicates to such a function (i.e., there is a primitive recursive function f such that $P(X) \equiv \lim_{y \to \infty} (X,y) = 1$, and such that for each X there are at most k+1 values of y for which $f(X,y) \neq f(X,y+1)$.

Proof. The primitive recursive sets can be enumerated in such a way that there is a uniform effective procedure for going from a gödel number (i.e., from s(i), if the sets in the enumeration in question are $W_{s(0)}, W_{s(1)}, \ldots$, where s(i) is a suitable primitive recursive function) to a decision method (primitive recursive characteristic function). But from the definition of Y_0 in the proof of Theorem 2, it is clear that Y_0 is primitive recursive whenever f is. Thus an enumeration of all the predicates of the form (A) with primitive recursive Y_0 contains all the predicates of the form (A) (for a given k) with a corresponding function f which is primitive recursive. And these predicates can be enumerated by a k-trial predicate (using the just-mentioned function s(i) instead of t(i) in the defintion of

P(i,x)). If this predicate corresponded to a primitive recursive f, so would P(i,i), and then by the argument of Lemma 3, so would $\overline{P}$(i,i). But this leads to the usual "Russell's Paradox" contradiction (given in full in the proof of the next theorem).

To prove the other half of the Corollary, we use the primitive recursiveness of the T predicate and the fact that the primitive recursive predicates are closed under bounded quantification to conclude that the "sequence of yesses and nos" constructed in the proof of the preceding theorem to show that P(i,x) is a k+1-trial predicate (i.e., that "with uniformity" the predicates of the form (A) are k + 1-trial) is primitive recursive. Replacing "yes" by 1 and "no" by 0 we then have the desired primitive recursive function.

Theorem 10. ("Hierarchy theorem" for $\sum_1^*$:) for each k, there is a k + 1-trial predicate which is not a k-trial predicate.

Proof. (The usual "diagonal argument":) The k + 1-trial predicate P(i,x) of the preceding theorem is such a predicate. For, if P(i,x) were k-trial so would P(i,i) be (since the k-trial predicates are evidently closed under substitutions), and hence so would $\overline{P}$(i,i) be (by Lemma 3). Then by Theorem 9 there would be an e such that $\{ i \mid \overline{P}(i,i) \} = \{ i \mid P(e,i) \}$. Setting e = i, we have $P(e,e) \equiv e \in \{ i \mid P(e,i) \} \equiv e \in \{ i \mid \overline{P}(i,i) \} \equiv \overline{P}(e,e)$, which is a contradiction.

Corollary. For k > 0, the k-trial predicates are closed under neither conjunction nor disjunction.

Proof. The k-trial predicates are closed under negation, and for $k \geq 1$ they include all r.e. predicates (since every r.e. predicate A can be written as $A \cup (\Lambda - \Lambda) \cup \ldots \cup (\Lambda - \Lambda)$). So, if for some k they were closed under either conjunction or disjunction, the k-trial predicates (with that fixed value of k) would contain all of Σ_1^*, contrary to Theorems 9 and 2.

Following Dekker, we shall call an infinite set "immune" if it has no infinite r.e. subset. Before leaving the class Σ_1^*, we give a new proof of

Theorem 11. (First proved by Markwald:) *if P ε Σ_1^*, then either P is not immune or $\overline{P}$ is not immune.*

Proof. If P ε Σ_1^*, then P is k-trial for some k by Theorem 2. Let $k > 0$ be the least k such that P is k-trial (for $k = 0$ the theorem is trivial), and let P be expressed in the form (A) by the method used in Theorem 2. We recall that:

(i) $Y_k \cup N_k$ = class of all X on which the machine "changes its mind" exactly k times ($f(X,y) \neq f(X,y+1)$ for exactly k values of y).

(ii) Y_k = subclass of $Y_k \cup N_k$ on which the answer is "yes" after the k-th "change of mind" ($f(X,a_k+1) = 1$).

(iii) N_k = subclass of $Y_k \cup N_k$ on which the answer is "no" after the k-th "change of mind" ($f(X,a_k+1) \neq 1$).

$Y_k \cup N_k$ must be infinite, since otherwise P would be (k-1)-trial, contrary to the choice of k[18]. Hence either Y_k or N_k must be infinite, and clearly $Y_k \subset P$ and $N_k \subset \overline{P}$. Thus either P or $\overline{P}$ is not immune. q.e.d.

We note that this is an example of a property of Σ_1^*-predicates which is not at all obvious on the basis of the

definition of $\sum_1^*$, but which is quite clear on the basis of Theorem 2.

Let us call a function g_P a modulus of convergence of f_P if, for all X,

$$f_P(X,y) = f_P(X, g_P(X)) \quad \text{whenever } y \geq g_P(X)$$

we have at once:

Theorem 12. If P is a trial and error predicate, then a[19] corresponding function f can have a recursive modulus of convergence if and only if P is recursive.

Proof. (Evident, since P could be defined in terms of the recursive functions f, g_P in the form

$$P(X) \equiv f(X,g_P(X)) = 1.)$$

For any f corresponding to a trial and error predicate P, let $h_f(X) =_{df}$ the number of values of y for which $f(X,y) \neq f(X,y+1)$. In analogy with the notion of "modulus of convergence" we will call any function g_f satisfying $g_f \geq h_f$ (for all X) a modulus of oscillation of the function f. We have:

Theorem 13. If P is a trial and error predicate, P can have a recursive modulus of oscillation even though P is not recursive. However, the "best possible" modulus (i.e., h_f) cannot be recursive unless P is.

Proof: Every k-trial predicate has by definition a constant modulus of oscillation, namely $g_f(X) = k$. Since the k trial predicates include the r.e. predicates (for $k \geq 1$), there is a predicate with a recursive modulus of oscillation which is not recursive. On the other hand, if h_f itself is recursive,

P can be defined in terms of f in the form:

$$P(X) \equiv f(X, a_{h_f(X)} + 1) = 1$$

where $a_{h_f(X)}$ is the $h_f(X)$th value of y, in order of magnitude, for which $f(X,y) \neq f(X,y+1)$.

It should be remarked that the predicates in $\Sigma_2 \cap \Pi_2$ which seem to naturally arise in metamathematics normally possess a primitive recursive modulus of oscillation. For example, the construction given in Kleene (cf. , p) of a $\Sigma_2 \cap \Pi_2$ -model for an arbitrary consistent formula of quantification theory actually leads to a model in which every predicate corresponds to an f with a very simple primitive recursive modulus of oscillation; likewise for predicate calculus with identity; and likewise for Markwald's result that there is a P in $\Sigma_2 \cap \Pi_2$ such that $P, \overline{P}$ are both immune. By contrast, the last result is false for Σ_1^*, as we proved above; the first is true for Σ_1^*, but difficult to prove (our Theorem 3); while the second (analogue of Theorem 3 for predicate calculus with identity) is still an open problem. Thus we may say that $\Sigma_2 \cap \Pi_2$-predicates which are not (or are not obviously) k-trial predicates for any k frequently occur in the literature; but not (as far as I know) predicates which are not obviously in the class of P for which there is at least one corresponding f with a primitive recursive modulus of oscillation. An example of this observation is

Theorem 14. The n-place predicates in Σ_1^* can be enumerated by an n+1-place predicate $P \in \Sigma_2 \cap \Pi_2$, whose corresponding f g can be expressed in terms of f in the form

$$g(x) = f(t_{g(x)}) + 1 \ldots$$

where $t_{g(x)}$ is the [illegible] value of [illegible] in place of equations.

[illegible]

It should be remarked that the condition [illegible] is [illegible] which is [illegible]

[illegible]

Theorem [illegible]. [illegible]

has a primitive recursive modulus of oscillation. (Sharpening the result of Mostowski, which results if the reference to the modulus of oscillation is deleted.)

Proof. It suffices to modify the proof of Theorem 9 by letting k vary when the enumeration of all the predicates of the form (A) is made. (This requires using an enumeration of all finite sequences, say as provided by the Chinese Remainder Theorem[20], instead of just the finite sequences of a given length.) The ith predicate of the form (A) will have $k+1$ as a (uniform) modulus of oscillation, as we saw; so it suffices to arrange the enumeration so that the "k" (in the normal form) can be primitive-recursively "read off" from the index i. (This is easily done: details are left to the reader.)

In view of the foregoing, it is natural to ask whether or not every $P \in \Sigma_2 \cap \Pi_2$ corresponds to at least one f with at least a general recursive modulus of oscillation. The negative answer to this question will be an immediate corollary to our final theorem, the proof of which also provides a considerable amount of information about other matters (e.g., the possibility, and effect on the modulus of oscillation, of using primitive recursive f instead of general recursive in the definition of trial and error predicate; the existence of a "normal form" for predicates with general recursive resp. primitive recursive moduli of oscillation):

Theorem 15. There is (for each n) a primitive recursive function $f(i,X,y)$ such that $\lim_{y \to \infty} f(i,X,y)$ always exists, and such that the $\Sigma_2 \cap \Pi_2$- predicate $\lim_{y \to \infty} f(i,X,y) = 1$ enumerates all the

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

<u>n-place predicates P in $\sum_2 \cap \prod_2$ for which there exists at least one corresponding f with a general recursive modulus of oscillation.</u>

<u>Proof</u>. (As before, we use the notations for the singulary case.) We will coordinate to $(q_i{*})_0$ (the ith function in $G^1 \cup F^2$) a primitive recursive function q_i' defined so that if, for fixed x, the successive values of $(q_i{*})_0$ are, say, 1,6,9 (this means $(q_i{*})_0(x,0) = 1, (q_i{*})_0(x,1) = 6, (q_i{*})_0(x,2) = 9$); then the successive values of q_i' may be 0,0,0,1,1,6,6,6,6,9,9....---i.e., q_i' (regarded as a sequence) consists of the same numbers as $(q_i{*})_0$, except for the initial zeros, and in the same order, but with (in general) more repetitions before a new value is taken on. Moreover, for each fixed x, the point at which the value $(q_i{*})_0(x,m)$ is taken on by q_i' will depend on the <u>computation</u> of $(q_i{*})_0(x,m)$: if the smallest y such that y is the gödel number of a proof that $(q_i{*})_0(x,m) = s$ is y_0, then the value $(q_i{*})(x,m-1)$ will be repeated in the sequence q_i' until the y_0th element has been passed (if necessary).

Formally:

(i) $q'(i,x,0) = 0$ (for all i,x)

(ii) $q'(i,x, m + 1) = 0$ if there is no $y \leq m+1$ such that y is a gödel number of a proof that $(q_i{*})_0(x,0)$ is defined.

(iii) $q(i,x,m+1) = (q_i{*})_0(x,0)$ if m+1 is the least gödel number of a proof that $(q_i{*})_0(x,0)$ is defined.

In this case, we say that m+1 x-corresponds to 0.

(iv) If m+1 x-corresponds to i, and no number $< m + 2$ is the gödel number of a proof that $(q_i{}^*)_0(x,i+1)$ is defined, then

$q'(i,x,m+2) = q'(i,x,m+1)$, and m+2 also x-corresponds to i.

(v) If m+1 x-corresponds to i and some number $\leq m+2$ is the gödel number of a proof that $(q_i{}^*)_0(x,i+1) = s$, for some s, then

$q'(i,x,m+2) = (q_i{}^*)_0(x,i+1)$, and m+2 x-corresponds to i+1.

Also, we put $q_i'(x,m) =_{df} q'(i,x,m)$. It is clear from the definition that q_i' is primitive recursive (noting that a simultaneous definition by primitive recursion of a function and a relation---such as our "x-corresponds to"--- can be always reduced to a simple definition by primitive recursion by well-known techniques) uniformly in i, and has the following properties:

(1) if $\lim_{y\to\infty} (q_i{}^*)_0(x,y)$ exists, so does $\lim_{y\to\infty} q_i'(x,y)$ and

$\lim_{y\to\infty} q_i'(x,y) = \lim_{y\to\infty} (q_i{}^*)_0(x,y)$; and

(2) Even if $(q_i{}^*)_0 \varepsilon F$, the function q_i' will be total (the "last" value of $(q_i{}^*)_0$ will simply be repeated forever, for each value of x, in the sequence $q_i'(x,0), q_i'(x,1),\ldots$).

Since every general recursive function is in $G \cup F$, and $\lim_{y\to\infty} (q_i{}^*)_0(x,y) = \lim_{y\to\infty} q_i'(x,y)$ whenever $\lim_{y\to\infty} (q_i{}^*)_0(x,y)$ exists, it is clear that the class of trial and error predicates is unchanged if we restrict the f to lie in the family $q_0', q_1', \ldots$

Also, we note that if $(q_i^*)_0$ is a total function with R(x) as a modulus of oscillation, then R(x) + 1 is a modulus of oscillation for q_i'(" + 1" because the 0 starting value can make one more possible value of y for which $q_i'(x,y) \neq q_i'(x,y+1)$)--- this is clear from the "x-corresponds to" relation between successive values of $(q_i^*)_0(x,y)$ and successive groups of repeated values of $q_i'(x,y)$. However, $\lim_{y \to \infty} q_i'(x,y)$ does not always exist. To rectify this, we now consider new functions $q_{i,j}(x,y)$, constructed from pairs q_i', $(q_j^*)_0$ (with first member in q_0', $q_1', \ldots,$ and second member in $G^1 \cup F^1$) as follows:

(i) $q(i,j,x,e) = 0$ if there is no $m \leq e$ such that m is the gödel number of a proof that $(q_j^*)_0(x) = s$, for some s

(ii) $q(i,j,x,y + m_0) = q_i'(x,y)$, if m_0 is the smallest number which is the gödel number of a proof that $(q_j^*)_0(x)=s$, for some s, and there are not more than s values of $m < y$ such that $q_i'(x,y) \neq q_i'(x, m+1)$

(iii) otherwise $q(i,j,x,y+1) = q(i,j,x,y)$

Also we put $q_{i,j}(x,y) =_{df} q(i,j,x,y)$.

We now maintain that (1) the sequence $q_{K(0),L(0)}$, $q_{K(1),L(1)}, \ldots$ consists of functions which all have a general recursive modulus of oscillation; and (2) if $P \in \Sigma_2 \cap \Pi_2$ and there is <u>any</u> corresponding $\underline{f}$ with a general recursive modulus of oscillation, then there is such an $\underline{f}$ in this family.

To prove (1), observe that if $(q_j{*})_0(x)$ is total, then, by the construction of $q_{i,j}$, $(q_j{*})_0$ is a modulus of oscillation for $q_{i,j}$. But if $(q_j{*})_0$ is not total, $(q_j{*})_0$ has a finite domain. In this case, $q_{i,j}(x,y) = 0$ for all y, except when x has one of a finite set of values. So 0 is a modulus of oscillation with finitely many exceptions, and on these exceptions there are still only finitely many values of y for which $q_{i,j}(x,y) \neq q_{i,j}(x,y+1)$. Hence there is a modulus of oscillation which is zero with finitely many exceptions, and a fortiori a recursive modulus of oscillation. To prove (2), observe that if $q_i'(x,y)$ has $(q_j{*})_0(x)$ as a modulus of oscillation [21] then $q_{i,j}(x,y)$ has the same limit and modulus of oscillation as $q_i'(x,y)$ uniformly in x. But every q_i' which has a recursive modulus of oscillation has some $(q_j{*})_0$ as a modulus, since all recursive functions are in $G \cup F$. Thus, setting

$$f(i,x,y) = q(K(i),L(i),x,y)$$

we have the theorem.

Corollary. The n-place predicates which correspond to at least one f with a primitive recursive modulus of oscillation are enumerated by a single n+1-place predicate with a corresponding f which has a general recursive modulus of oscillation.

Proof. It is well known that the n-place primitive recursive functions are enumerated by a single n+1-place general recursive function. Hence, by the Iteration Theorem (cf. n.) there exists a singulary primitive recursive function s such that $q^n_{s(0)}, q^n_{s(1)}, \ldots$ are the n-place primitive recursive functions.

Since $(q^n_{s(j)}{}^*)_0 = q^n_{s(j)}$ is a modulus of oscillation for $q^n_{i,s(j)}$ (because all primitive recursive functions are total), and since f can be restricted to lie in the family $\{q_0', q_1', \ldots\}$ (cf. n.   ), we have at once that the predicate

$$\lim_{y \to \infty} q^n(K(i),\ s(L(i)), X, y) \text{ has the desired}$$

properties. (Here we write the superscript "n", because we have explicitly indicated the general case by writing "X" for "x".)

Corollary B. There exists a predicate which corresponds to an f with a general recursive modulus of oscillation, but not to any f with a primitive recursive modulus of oscillation.

Proof. The predicate constructed in the proof of the preceding corollary has this property, by the "diagonal argument".

Corollary C. There is a trial and error predicate which does not correspond to any f with even a general recursive modulus of oscillation.

Proof. The predicate constructed in the proof of Theorem 15 (namely, $\lim_{y \to \infty} f(i,x,y) = 1$) has this property, by the "diagonal argument".

FOOTNOTES

*) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF49(638)-777. Reproduction in whole or in part is permitted for any purpose of the United States Government.

1) For going from a finite sequence to an infinite sequence (with repetitions) cf. the proof of Theorem 3, below. Going in the other direction is trivial: it suffices to instruct the machine that it is to "print out" an answer only when it is different from the previous answer.

2) For a definition of this concept see the beginning of § 4.

3) Intuitively, g(x) is a "modulus of oscillation" if, for all x, the machine never changes its mind more than g(x) times given x as "input". (The formal definition is in § 5.) That there be a recursive modulus of oscillation is evidently a very lenient requirement on a trial and error predicate.

4) An expression (Ex)(y)R, where R is a recursive predicate, is called a Σ_2-expression here, and (x)(Ey)R is called a Π_2-expression. Predicates that can be expressed in both these forms form the class $\Sigma_2 \cap \Pi_2$. (Cf. [1], ch. 9; Davis, however, uses "P" and "Q" where we use Σ and Π.)

5) Cf. [3].

6) We use K* to denote the closure of a class of predicates K under truth functions. In particular, Σ_1^* is the smallest class of predicates containing the r.e. predicates and closed under truth-functions.

7) Cf. [2], p. 394, Theorem 35.

8) In [4].

9) Cf. [7]. This system is chosen because it is strong enough so that all recursive functions are formally reckonable in it, but weak enough so that its consistency admits of a constructive proof.

10) This is, of course, just another way of writing

$$G(x_1,\ldots,x_n) \equiv (Ey_1,\ldots,y_n)(F(y_1,\ldots,y_n) \& x_1 \in R(y_1) \& \ldots \& x_n \in R(y_n)).$$

11) Since we are considering formulas with only one predicate letter, we can identify a model with a pair $< A,B >$ such that the formula is true when the individual variables range over A and the predicate letter is interpreted as standing for B.

12) Every consistent formula has such a model, by the theorem cited in n.7.

13) Here and in the sequel, J is the widely used (see [1], pp. 43-45) recursive mapping of pairs of integers onto (different) integers. It has the property that every number $x = J(y,z)$ for uniquely determined y,z (usually written $y = K(x)$, $z = L(x)$), where all three functions J,K,L are primitive recursive.

14. The symbol λ_x may be read "the function whose value for any x is".

15. We assume, of course, that some normal form for the statements "$(y)_{<x} q_i(y)$ is defined" is adopted in the notation of first order arithmetic, such that the gödel number of such a statement is a recursive function of i and x, and such that

Robinson's arithmetic is complete and correct for statements of this form. (Cf. [7], [6].)

16) I.e., the function C_P defined by $C_P(X) = 1 \equiv P(X)$ and $C(X) = 0 \equiv \overline{P}(X)$.

17) This is Kleene's "S^m_n" Theorem. Cf. [2], p. 342, [1], [6].

18) Observing that if the machine "changes its mind" not more than k-1 times, except on finitely many X, the program can always be changed on these finitely many X to show that P is a R-1-trial predicate.

19) "A corresponding f" means, of course, an f in terms of which P can be defined as in § 1 of this paper.

20) Cf. [1], appendix.

21. Here use has been made of the fact that one may restrict f to be in the family $\{q_0', q_1', \ldots\}$ without altering the modulus of oscillation by more than + 1, and hence without altering such properties as having a general recursive modulus of oscillation.

REFERENCES

[1] MARTIN DAVIS, Computability and Unsolvability, New York, 1958.

[2] STEPHEN COLE KLEENE, Introduction to metamathematics, New York, 1952.

[3] ANDRZEJ MOSTOWSKI, A formula with no recursively enumerable model. Fundamenta Mathematica, vol. 42 (1955), pp. 125-140.

[4] HILARY PUTNAM, Arithmetic models for consistent formulas of quantification theory, Journal of Symbolic Logic, vol. 22 (1957), pp. 110-111.

[5] HILARY PUTNAM, Decidability and essential undecidability, Journal of Symbolic Logic, vol. 22 (1957), pp. 39-54.

[6] RAYMOND SMULLYAN, Theory of formal systems, Princeton, 1961.

[7] ALFRED TARSKI, ANDRZEJ MOSTOWSKI, and RAPHAEL M. ROBINSON, Undecidable theories, Amsterdam, 1953.

DISTRIBUTION LIST AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
MATHEMATICAL SCIENCES DIRECTORATE

(ONE COPY UNLESS OTHERWISE NOTED)

ALABAMA

Commander
Army Rocket & Guided Missile Agency
ATTN: ORDXR-OTL
Redstone Arsenal, Alabama

BELGIUM

Commander (3 Copies)
European Office, ARDC
47 Rue Cantersteen
Brussels, Belgium

CALIFORNIA

Applied Mathematics & Statistics Laboratory
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander
Air Force Flight Test Center
ATTN: Technical Library
Edwards Air Force Base
California

The Rand Corporation (2 Copies)
Technical Library
1700 Main Street
Santa Monica, California

Commander
1st Missile Division
ATTN: Operations Analysis Office
Vandenburg Air Force Base
California

CONNECTICUT

Department of Mathematics
Yale University
New Haven, Connecticut

DISTRICT OF COLUMBIA

Office of Naval Research (2 Copies)
Department of the Navy
ATTN: Code 432
Washington 25, D.C.

Director
Department of Commerce
Office of Technical Services
Washington 25, D.C.

Administrator (6 Copies)
National Aeronautics and Space Administration
ATTN: Documents Library
1520 H Street, N.W.
Washington 25, D.C.

Library
National Bureau of Standards
Washington 25, D.C.

Data Processing Systems Division
National Bureau of Standards
ATTN: Mr. Russel A. Kirsch
Washington 25, D.C.

Applied Mathematics Division
National Bureau of Standards
Washington 25, D.C.

Headquarters, USAF
Assistant for Operations Analysis
Deputy Chief of Staff Operations, AFOOA
Washington 25, D.C.

Commander (2 Copies)
Air Force Office of Scientific Research
ATTN: SRM
Washington 25, D.C.

Director
U. S. Naval Research Laboratory
ATTN: Library
Washington 25, D.C.

National Science Foundation
Program Director for Mathematical Sciences
Washington 25, D.C.

Commander, AFRD (2 Copies)
ATTN: Technical Library
Washington 25, D.C.

Canadian Joint Staff
ATTN: DRB/DSIS
2450 Massachusetts Avenue, N.W.
Washington, D.C.

ILLINOIS

Department of Mathematics
Northwestern University
Evanston, Illinois

Laboratories for Applied Sciences
University of Chicago
Museum of Science and Industry
ATTN: Library, W-305
Chicago 37, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

INDIANA

Department of Mathematics
Purdue University
Lafayette, Indiana

MARYLAND

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Director
National Security Agency
ATTN: Dr. H. H. Campaign
Fort George G. Meade
Maryland

MASSACHUSETTS

Department of Mathematics
Harvard University
Cambridge 38, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge 38, Massachusetts

Commander
Detachment 2, AFRD
ATTN: Technical Library
L. G. Hanscom Field
Bedford, Massachusetts

MICHIGAN

Department of Mathematics
Wayne State University
Detroit 1, Michigan

MINNESOTA

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

MISSOURI

Department of Mathematics
Washington University
St. Louis 8, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

NEBRASKA

Commander
Strategic Air Command
ATTN: Operations Analysis
Offutt Air Force Base
Omaha, Nebraska

NEW JERSEY

The James Forrestal Research Center Library
Princeton University
Princeton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princeton, New Jersey

Commanding General
Signal Corps Engineering Laboratory
ATTN: SIGFM/EL-RPO
Ft. Monmouth, New Jersey

NEW MEXICO

Commander
Air Force Missile Development Center
ATTN: Technical Library, HDOI
Holloman Air Force Base
New Mexico

Commander
Air Force Special Weapons Center
ATTN: Technical Library, SWOI
Kirtland Air Force Base
Albuquerque, New Mexico

NEW YORK

Professor J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

Institute for Mathematical Sciences
New York University
ATTN: Professor M. Kline
25 Waverly Place
New York 3, New York

Institute for Aeronautical Sciences
ATTN: Librarian
2 East 64th Street
New York 16, New York

NORTH CAROLINA

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

Office of Ordnance Research (2 cys.)
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

OHIO

P.O. Box AA
Wright-Patterson Air Force Base
Ohio

Commander
Wright Air Development Division
ATTN: WCOSI
Wright-Patterson Air Force Base
Ohio

Commander
Aeronautical Research
Laboratories
ATTN: Technical Library
Wright-Patterson Air Force Base
Ohio

USAF Institute of Technology
Library (2 copies)
ATTN: MCLI-ITLIB
Building 125, Area B
Wright-Patterson Air Force Base
Ohio

PENNSYLVANIA

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

TENNESSEE

AEDC Library
ARO, Inc.
Arnold AF Station
Tennessee

U.S. Atomic Energy Commission
Technical Information Service
Extension
P.O. Box 62
Oak Ridge, Tennessee

TEXAS

Applied Mechanics Reviews (2 Copies)
Southwest Research Institute
8500 Culebra Road
San Antonio 6, Texas

Department of Mathematics
Rice Institute
Houston, Texas

VIRGINIA

Armed Services Technical Information
Agency (10 Copies)
ATTN: TIPDR
Arlington Hall Station
Arlington 12, Virginia

WISCONSIN

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Mathematics Research Center,
U.S. Army
ATTN: R. E. Langer
University of Wisconsin
Madison, Wisconsin